Grids for effcient all sky search of white dwarf binaries in Mock LISA Data Challenge

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Abstract

We present construction of the 3 and 4 dimensional grids in the parameter space for all sky-search of gravitational wave signals from white dwarf binaries with LISA data. The 3 dimensional grid is for search of frequency and the sky position of the source and the 4 dimensional grid includes the spin down parameter. The grid solves the covering problem in the parameter space with the constraint that nodes of the grid coincide with Fourier frequencies (multiples of the inverse of the observation time).This allows the use of the fast Fourier transform (FFT) in the evaluation of the optimal statistic and greatly speeds up the search.

Introduction

We consider the problem of construction of the grid in the parameter space for white dwarf binary systems. We sketch a search method that enables fast evaluation of the $\mathcal{F}$-statistic by using the FFT algorithm. The metric defined on the parameter space is flat but grid points are constrained in one direction. We present two constructions of nearly optimal lattices in any number of dimensions that in a good approximation satisfy the constraint. We illustarate these procedures for 3 and 4-dimensional parameter spaces.

Search algorithm

The response of the space-based detector LISA to a gravitational wave can be written[1] as a linear combination of 4 functions $h^{(k)}$ depending on the time and a set of so called *intrinsic* parameters ξ^μ with constant coefficients $a^{(k)}$ called *extrinsic* parameters:

$$h(t; a^{(k)}, \xi^\mu) = \sum_{k=1}^{4} a^{(k)} h^{(k)}(t; \xi^\mu). \qquad (1)$$

In the maximum-likelihood search method the estimators $(\hat{a}^{(k)}, \hat{\xi}^\mu)$ are found by maximizing the log likelihood function with respect to parameters $(a^{(k)}, \xi^\mu)$. By solving explicitly the maximum likelihood equation for the extrinsic parameters one can define the so called $\mathcal{F}$-statistic and reduce the search to the intrinsic parameters space. For a signal from white-dwarf binaries burried in a stationary Gaussian noise and for intrinsic parameters defined by $\xi^\mu = (\omega, \dot{\omega}, \beta, \lambda)$, where β and λ are the latitude and the longitude of the source, the detection statistic $\mathcal{F}$ consists of integrals of the form

$$\int y(t) m(t; \omega, \beta, \lambda) \exp\left[i\phi_{mod}(t; \omega, \dot{\omega}, \beta, \lambda) \exp i\omega t\right], \qquad (2)$$

where $y(t)$ are the noisy data and m, ϕ_{mod} are amplitude and phase modulation functions. As the modulation functions m and ϕ_{mod} depend on the frequency ω we cannot apply the FFT algorithm directly in calculation of $\mathcal{F}$-statistic. In order to do that we first analyse the data in narrow band over which the slowly varying modulation function m is evaluated at the mid frequency of the band and, second, we introduce a new linear parametrization of the phase

$$\phi_{mod} = \frac{1}{2}\dot{\omega}t^2 + A\cos\Omega t + B\sin\Omega t,$$

where frequency is absorbed in the new parameters A and B and where $\Omega = 2\pi/\text{yr}$.

A linear model

In order to construct the grid of templates we use the following linear signal model

$$h(t) = A_0 \cos\left(\omega t + \frac{1}{2}\dot{\omega}t^2 + A\cos\Omega t + B\sin\Omega t + \phi_0\right) \qquad (3)$$

with extrinsic (A_0, ϕ_0) and intrinsic $(\omega, \dot{\omega}, A, B)$ parameters.
We shall define a grid in the intrinsic parameter space such that for any signal defined by ξ_s there exists a grid point ξ_g such that

$$E_{\xi_s}[\mathcal{F}(y; \xi_g)] \geq C$$

for some fixed value C. This leads to the *reduced Fisher matrix* $\tilde{\Gamma}$ which for (ω, A, B) and $(\omega, \dot{\omega}, A, B)$ parameter space takes the form (in dimensionless units)

$$\tilde{\Gamma} = \begin{pmatrix} \frac{1}{12} & 0 & -\frac{1}{2\pi n} \\ 0 & \frac{1}{2} & 0 \\ -\frac{1}{2\pi n} & 0 & \frac{1}{2} \end{pmatrix}, \quad \tilde{\Gamma} = \begin{pmatrix} \frac{1}{12} & \frac{1}{24} & 0 & -\frac{1}{2\pi n} \\ \frac{1}{24} & \frac{1}{45} & \frac{1}{4\pi^2 n^2} & -\frac{1}{4\pi n} \\ 0 & \frac{1}{4\pi^2 n^2} & \frac{1}{2} & 0 \\ -\frac{1}{2\pi n} & -\frac{1}{4\pi n} & 0 & \frac{1}{2} \end{pmatrix}$$

where integer n is the number of years.

Covering problem with constraints

We want to distribute the points of the grid in such a way that the distance defined by the metric $\tilde{\Gamma}$ from any point of the parameter space to the nearest node of the grid is not larger than some fixed value r.
The problem of constructing the grid in the parameter space is then equivalent to the problem of covering d-dimensional space with equal overlapping spheres of a given radius. One is interested in an optimal covering, i.e. the one having smallest possible number of grid points per unit volume. This defines the so called *covering problem.* The *thickness* Θ of a covering is defined [2] as the average number of spheres that contain a point of the space.

The optimal covering would have minimal possible thickness. We consider only lattice coverings.
We want to calculate the integral (2) using the FFT algorithm. For this reason we would like the nodes of the grid to coinside with Fourier frequencies: $\Delta\omega, 2\Delta\omega, 3\Delta\omega, \ldots$ for some fixed frequency resolution $\Delta\omega$. It imposes the condition that one of the basis vectors of the lattice has a fixed length

$$|\vec{v}_0| = \sqrt{\tilde{\Gamma}\left[(\Delta\omega, 0, \ldots, 0), (\Delta\omega, 0, \ldots, 0)\right]}$$

and forbidds an immediate use of the general results of the theory of lattice coverings. Instead one can formulate the *covering problem with constraint*: to find the thinnest lattice covering of the d-dimensional space with spheres of radius r and one of the basis vectors of the lattice having fixed length $|\vec{v}_0|$.
As far as we know the general solution to the problem is not known. We present two constructions of lattices in any number of dimensions that in a good approximation satisfy the constraints. They can be viewed as possible solutions to the problem and may serve as starting point for alternative constructions.

Lattices

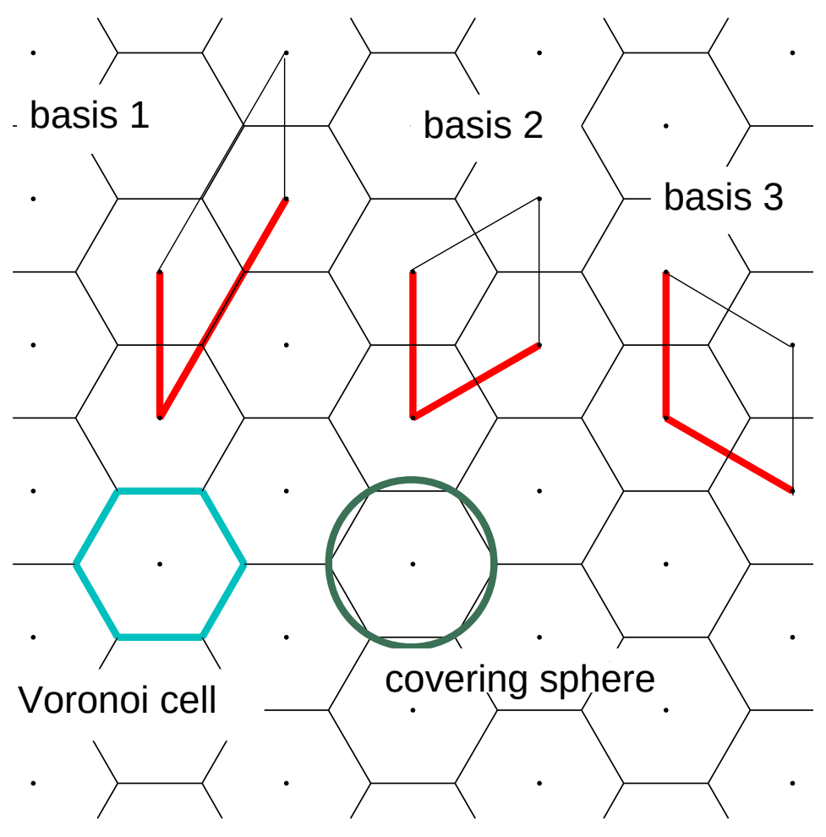
A *lattice*[2] Λ is a discrete subset of $\mathbf{R}^d$. Any lattice has a *basis* $b = \{\vec{b}_1, ..., \vec{b}_d\}$ of linearly independent vectors such that the lattice is the set of all linear combinations of $\vec{b}_i$'s with integer coefficients:

$$\Lambda(b) = \left\{ \sum_{i=1}^{d} c_i \vec{b}_i : \quad c_i \in \mathbf{Z}, \quad i = 1, 2, \ldots, d \right\}. \qquad (4)$$

The parallelotope consisting of points $c_1\vec{b}_1 + \ldots + c_d\vec{b}_1$ with $0 \leq c_i < 1$ is a *fundamental parallelotope* and is an example of an *elementary cell*, that is the building block containing one lattice point which tiles the whole $\mathbf{R}^d$ by translations by lattice vectors. Any lattice has infinitely many bases and elementary cells but the volume of each elementary cell is unique for a lattice.
The *Voronoi cell* around any point $\vec{v}$ of Λ is the set of vectors $\vec{x}$ of $\mathbf{R}^d$ which are closer to $\vec{v}$ than to any other lattice vector:

$$V(\vec{v}) = \{\vec{x} : \quad |\vec{x} - \vec{w}| \geq |\vec{x} - \vec{v}| \quad \text{for all} \quad \vec{w} \in \Lambda\}. \qquad (5)$$

All Voronoi cells of a given lattice are congruent convex polytopes and are another examples of elementary cells sometimes refered to as Wigner-Seitz cells or Brillouin zones.



In general for any discrete set of points $\mathcal{S} = \{\vec{s}_1, \vec{s}_2, \ldots\}$ in $\mathbf{R}^d$ the *covering radius* R of $\mathcal{S}$ is defined as the least upper bound for any point of $\mathbf{R}^d$ to the closest point $\vec{s}_i$:

$$R(\mathcal{S}) = \sup_{\vec{x} \in \mathbf{R}^d} \inf_{\vec{s} \in \mathcal{S}} |\vec{x} - \vec{s}|.$$

Then spheres of equal radius r centered at the points $\vec{s}_i$ will cover $\mathbf{R}^d$ only if $r \geq R$.
For the lattice Λ having Voronoi cells congruent to polytope $V(\vec{v})$, where $\vec{v}$ is any of the lattice points, the covering radius is the circumradius of $V(\vec{v})$ i.e. the largest distance between $\vec{v}$ and the vertices of $V(\vec{v})$.
The thickness of the lattice covering is given by:

$$\Theta(\Lambda) = \frac{\text{volume of } d\text{-dimensional sphere of radius } R(\Lambda)}{\text{volume of the elementary cell of } \Lambda} \qquad (6)$$

The thinnest *lattice* coverings are known in dimensions up to 5. They are defined by the so called Voronoi's principal lattices of the first kind, A_d^* [2]. In 3 and 4 dimensions they have thickness $\Theta(A_3^*) = 1.4635$ and $\Theta(A_4^*) = 1.7655$.

References

[1] P. Jaranowski, A. Królak ”Gravitational-wave data analysis. Formalism and simple applications: the Gaussian case”, *Living Rev. Relativity*, **8**, (2005), 3 (cited on 6 June 2008): http:/www.livingreviews.orglrr-2005-3

[2] J.H. Conway, N.J.A. Sloane *S*phere Packings, Lattices and groups (Springer-Verlag, New York, 1993)

Procedure I

We search for a lattice $\Lambda(w')$ with lattice basis w' satisfying two constraints that can be expressed as

$$R(\Lambda(w')) = r, \text{ and } w' = \left\{\vec{v}_0, \vec{w}'_1, \ldots, \vec{w}'_{d-1}\right\} \qquad (7)$$

We consider an orthonormal basis $\{\vec{w}_1, \ldots, \vec{w}_{d-1}\}$. As an initial approximation for the lattice we take $\Lambda(w)$ where $w = \{\vec{v}_0, \vec{w}_1, \ldots, \vec{w}_{d-1}\}$. The construction of $\Lambda(w')$ is based on the following algorithm **AI**.

```
Algorithm AI
Input: $w = \{\vec{v}_0, \vec{w}_1, \ldots, \vec{w}_{d-1}\}$.
Output: $w' = \left\{\vec{v}_0, \vec{w}'_1, \ldots, \vec{w}'_{d-1}\right\}$, $R(\Lambda(w')) \approx r$.
AI1. $w' = w$.
AI2. Repeat
AI3. $w' = \left\{\vec{v}_0, c\vec{w}'_1, \ldots, c\vec{w}'_{d-1}\right\}$, where $c = \frac{r}{R(\Lambda(w'))}$
AI4. Until $R(\Lambda(w')) \approx r$.
AI5. Return $w'$.
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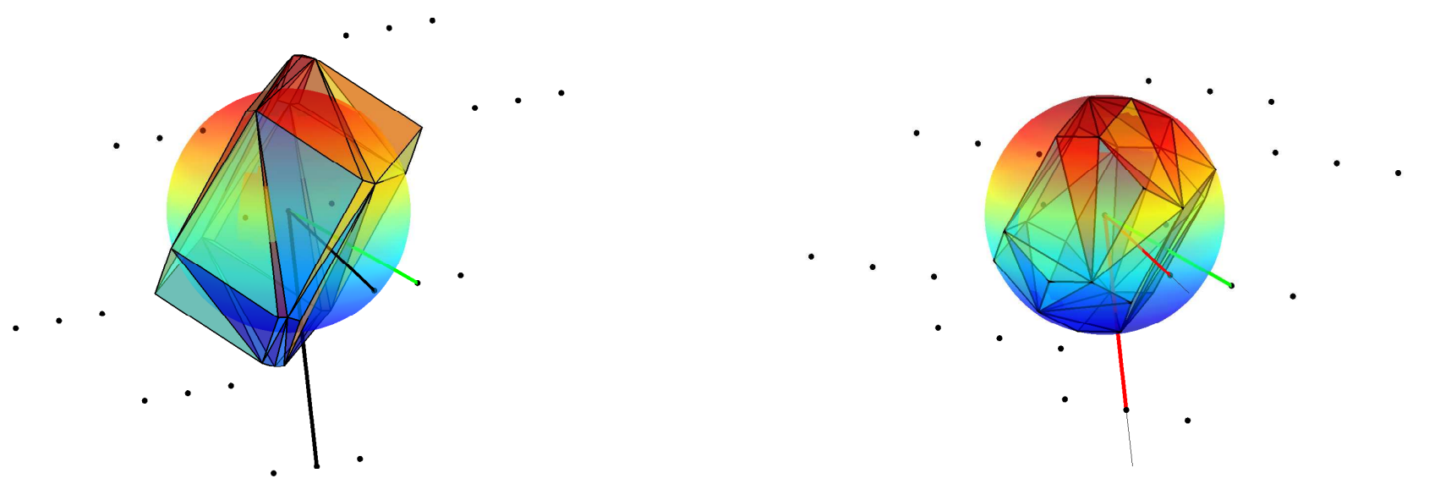

The Procedure I starts with an arbitrary orthonormal $(d-1)$-dimensional basis $\{\vec{w}_1, \ldots, \vec{w}_{d-1}\}$. To construct an optimal constrained lattice one repeats the algorithm **AI** $[\{\vec{v}_0, \vec{w}_1, \ldots, \vec{w}_{d-1}\}]$ with $\vec{v}_0$ having different orientations with respect to $\{\vec{w}_1, \ldots, \vec{w}_{d-1}\}$ and chooses w' having the smallest thickness.

Procedure II

The idea is to shrink an optimal lattice as slightly as possible such that one of the basis vectors of the resulting lattice coincides with the constraint vector $\vec{v}_0$.
We denote by $\vec{l}(\Lambda)$ the smallest vector of Λ satisfying $|\vec{v}_0| \leq |\vec{l}|$ and define the algorithm **AII**.

```
Algorithm AII
Input: Lattice $\Lambda$; vector $\vec{v}_0$.
Output: Lattice $\Lambda'$; $\vec{l}(\Lambda') = \vec{v}_0$.
AII1. Find $\vec{l}(\Lambda)$.
AII2. Contract $\Lambda$ along $\vec{l}(\Lambda)$ to obtain $\Lambda_c$.
AII3. Rotate $\Lambda_c$ to obtain a lattice $\Lambda_{rc}$ with $\vec{l}(\Lambda_{rc}) = \vec{v}_0$.
AII5. Return $\Lambda' = \Lambda_{rc}$.
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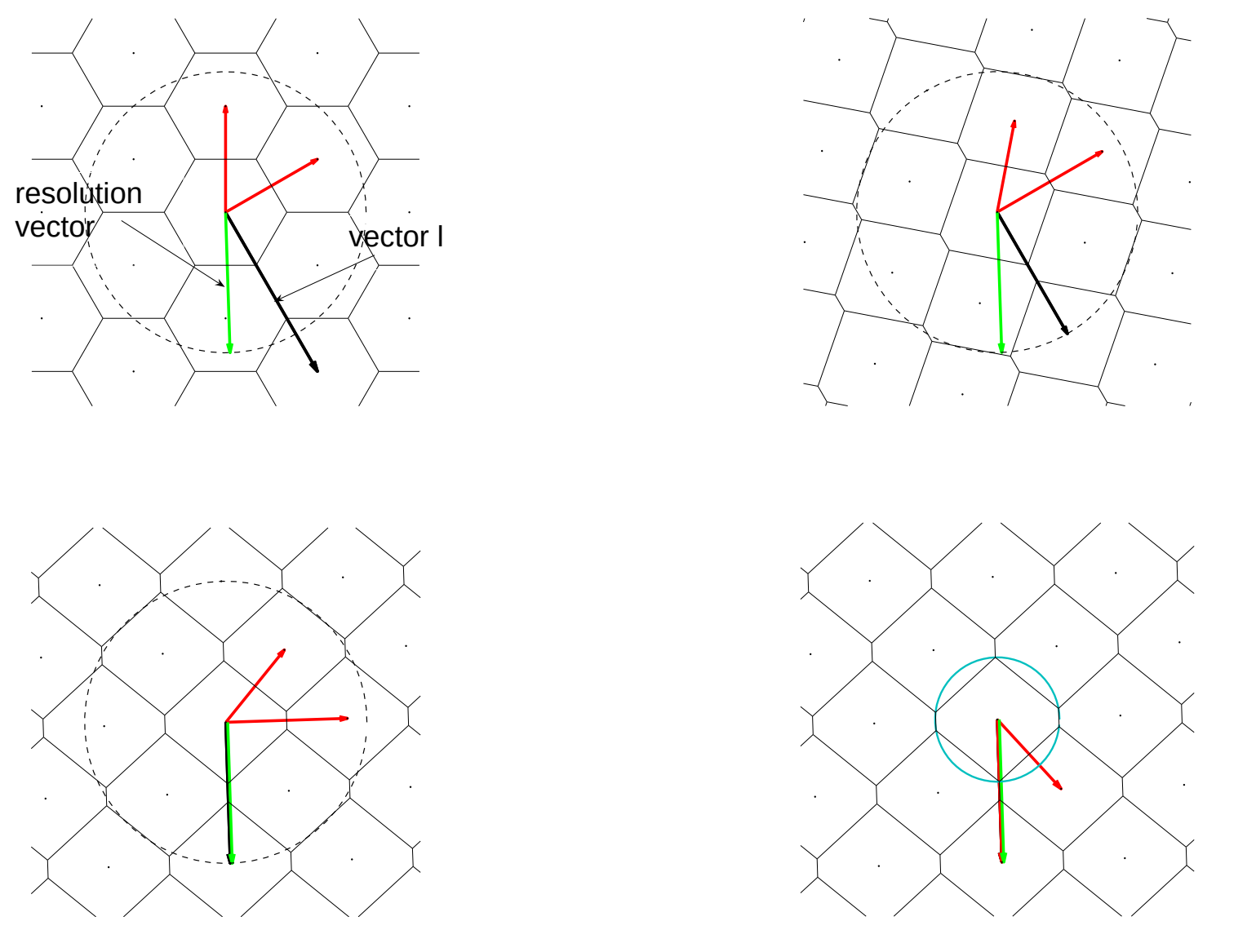
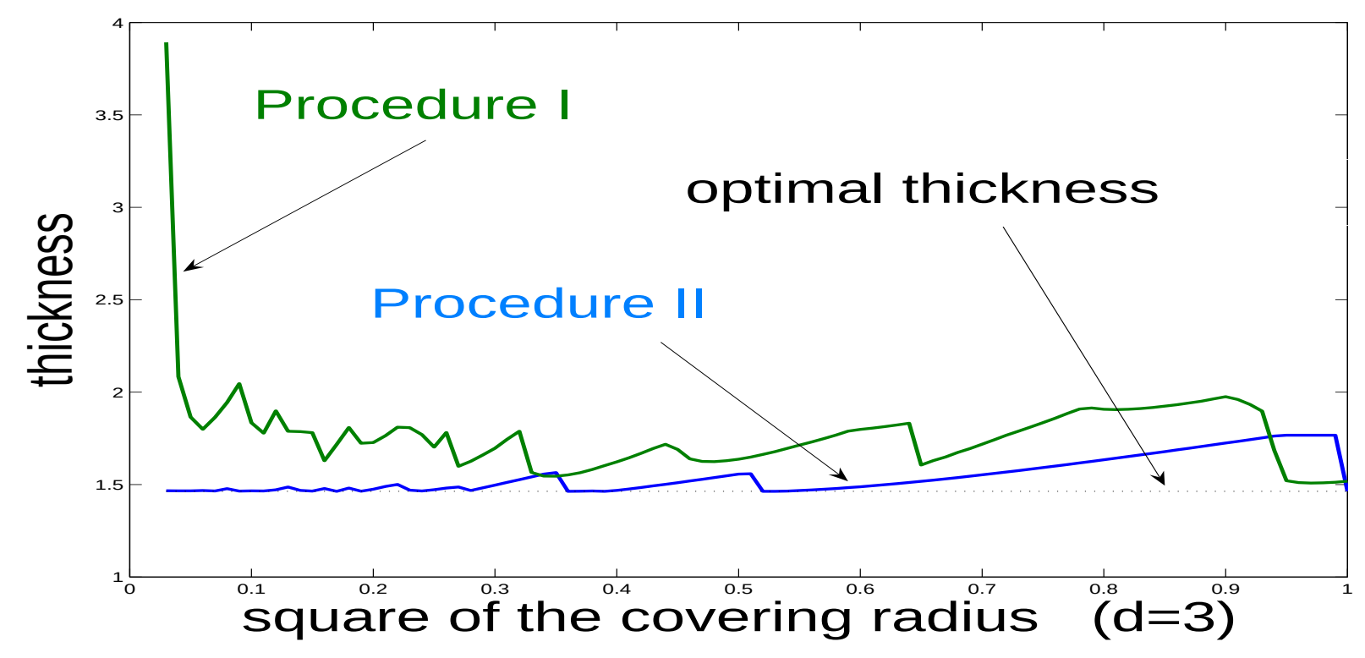
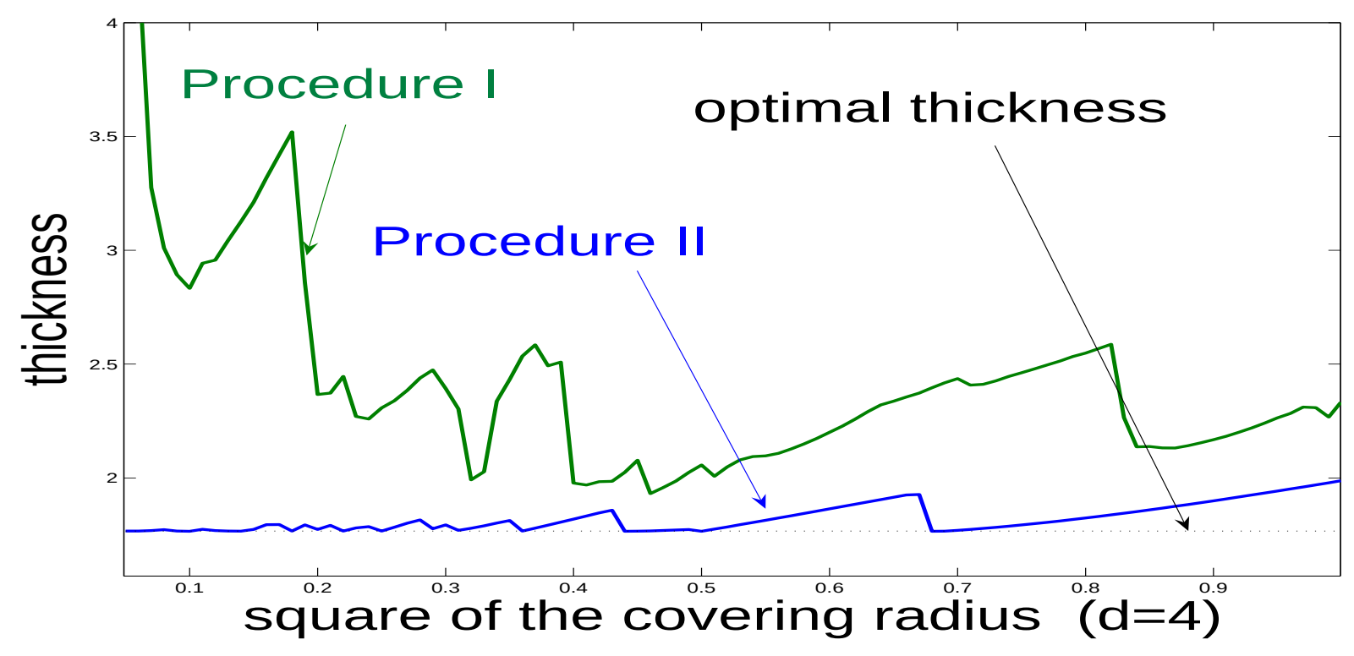


The optimal constrained lattice is obtained by application of the algorithm **AII** to an optimal (unconstrained) lattice Λ with a covering radius r_i such that $R(\Lambda') = r$.

Results

We present the results of the application of the two procedures to the model (3) in 3 and 4 dimensions with the resolution vector $\vec{v}_0 = (2\pi, 0, \ldots)$ and $n = 2$.





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