

Abstract

One of the primary candidate sources of gravitational wave signals falls under the category of large mass binary systems. In particular, extreme mass ratio inspiral (EMRI) systems such as that of a compact solar mass object (mass m) inspiralling into a black hole of 1,000 to 100,000,000 solar masses (mass M) are expected to be detectable by LISA out to Gpc distances. In modelling such a system, the smaller mass is seen to exert a force on itself which we call the self-force. The computation of this self-force is of fundamental importance to the accurate calculation of the orbital evolution of such binary systems and hence to the prediction of the gravitational radiation waveform.

Formal solutions of the self-force problem have been found whereby the solution is expressed in terms of an integral of the retarded Green's function over the entire past world-line of the small black hole. Although in principle this formal solution gives the desired result, in practice the calculation of the Green's function poses a formidable challenge. We present a matched expansion approach to calculating the self-force, whereby it is divided into a contribution from the recent past (the quasi-local region) and a contribution from the more distant past. In particular, we focus on calculating the contribution from the quasi-local region and present recent results [1].

Extreme Mass Ratio Inspirals and the Self-Force

As a prerequisite to the calculation of the waveform templates for binary system gravitational wave sources, it is necessary to model the orbital evolution of the system. In the case of extreme mass ratio systems, we can do so by making the approximation that the smaller mass is travelling in the background space-time of the larger mass. However, the compact nature of the smaller mass means that it itself will distort the curvature of the space-time in which it is moving to a non-negligible extent. As a result, it does not follow a geodesic of the background space-time of the larger mass, but rather, it follows a geodesic of the total space-time of both masses. In this sense the motion of the smaller mass is affected by the presence of its own mass, i.e. the smaller mass is seen to exert a force on itself. This is called the self-force.

Because of this non-linear behavior, it is extremely difficult, if not impossible, to exactly solve the equations of motion of such a system. However, due to the extreme mass ratio involved, the deviation of the smaller black hole's trajectory from the background geodesic is small over sufficiently small time-scales (less than the natural length-scale of the background). In this case, to linear order in m , we can perturbatively model the deviation from the background as a spin-2 field generated by the particle and living on the background space-time. This field is then seen to couple to the smaller black hole to cause the self-force.

In the present work, we will focus not on the gravitational self-force described thus far, but rather on the analogous but simpler scalar self-force. Instead of considering a smaller mass, m , to be generating a gravitational field, we consider a particle with scalar charge, q . This charge generates a massless scalar field to which it couples, resulting in the scalar self-force. Otherwise, we leave the problem exactly as posed above. In making this change, the resulting analysis remains extremely similar to the gravitational case, but the algebraic details of the calculations become simpler. This will allow us to explore the problem and develop techniques which we will at a later date apply to the physically more interesting gravitational case.

Matched Expansion Approach to the Self-Force

Consider a particle of scalar charge q and mass m travelling in a curved background space-time. The scalar field generated by the particle interacts with the particle, effectively causing a force, called the scalar self-force. This self-force may be expressed as a sum of local and non-local parts [2]:

$$f^\mu(x) = q^2 \left(\frac{1}{3} (\dot{a}^\mu - a^2 u^\mu) + \frac{1}{6} (R^{\mu\alpha} u_\alpha + R_{\alpha\beta} u^\alpha u^\beta u^\mu) + \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\tau-\epsilon} \nabla^\mu G_{ret}(x, x') d\tau' \right)$$

where u^α is the 4-velocity of the particle, a^α is the 4-acceleration, G_{ret} is the retarded scalar Green's function and $R_{\alpha\beta}$ is the Ricci tensor. In this form, all local terms are given explicitly, so their calculation can be carried out immediately without posing any real difficulty. Our task is therefore to elucidate the non-local integral of the gradient of the retarded Green's function over the entire past world-line of the particle. We do so by splitting the integral in two parts:

$$\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\tau-\epsilon} \nabla^\mu G_{ret} d\tau' = \lim_{\epsilon \rightarrow 0} \int_{\tau-\Delta\tau}^{\tau-\epsilon} \nabla^\mu G_{ret} d\tau' + \int_{-\infty}^{\tau-\Delta\tau} \nabla^\mu G_{ret} d\tau'$$

where $\tau-\Delta\tau$ is a matching point, chosen so that both integrals may be evaluated separately using different means and subsequently combined to obtain an expression for the total self-force. We call the first integral the quasi-local part of the self-force and the remainder of our discussion will focus on its computation.

The retarded Green's function in this integral is a function of the particle's current point, x , and a point x' on its past world-line. Provided x and x' are sufficiently close (i.e. provided $x'(\Delta\tau)$ doesn't lie too far along the past world-line), it is given by the Hadamard form:

$$G_{ret} = \theta_-(U\delta(\sigma) - V\theta(-\sigma))$$

where $\theta_-(x, x')$ is analogous to the Heaviside step-function (i.e. 1 when x' is in the causal past of x , 0 otherwise), $\delta(\sigma(x, x'))$ is the standard Dirac delta function, $\sigma(x, x')$ is the Synge world function (i.e. half the square of the geodesic distance between x and x') and $U(x, x')$ and $V(x, x')$ are symmetric bi-scalars having the benefit that they are regular for $x' \rightarrow x$.

The first term involving $U(x, x')$ - called the *direct* part - does not contribute to the self-force. This can be understood by the fact that the presence of $\theta_-(x, x')\delta(\sigma(x, x'))$ means that it is only non-zero on the past light-cone of the particle. However, since the quasi-local integral is totally internal to the past light-cone, this term does not contribute. The second term is known as the *tail*, and it is responsible for the self-force. As a result the expression for the quasilocal contribution to the self-force becomes:

$$f_{QL}^a = -q^2 \int_{\tau-\Delta\tau}^{\tau} \nabla^a V(x, x') d\tau'$$

and our task is to find an expression for $V(x, x')$.

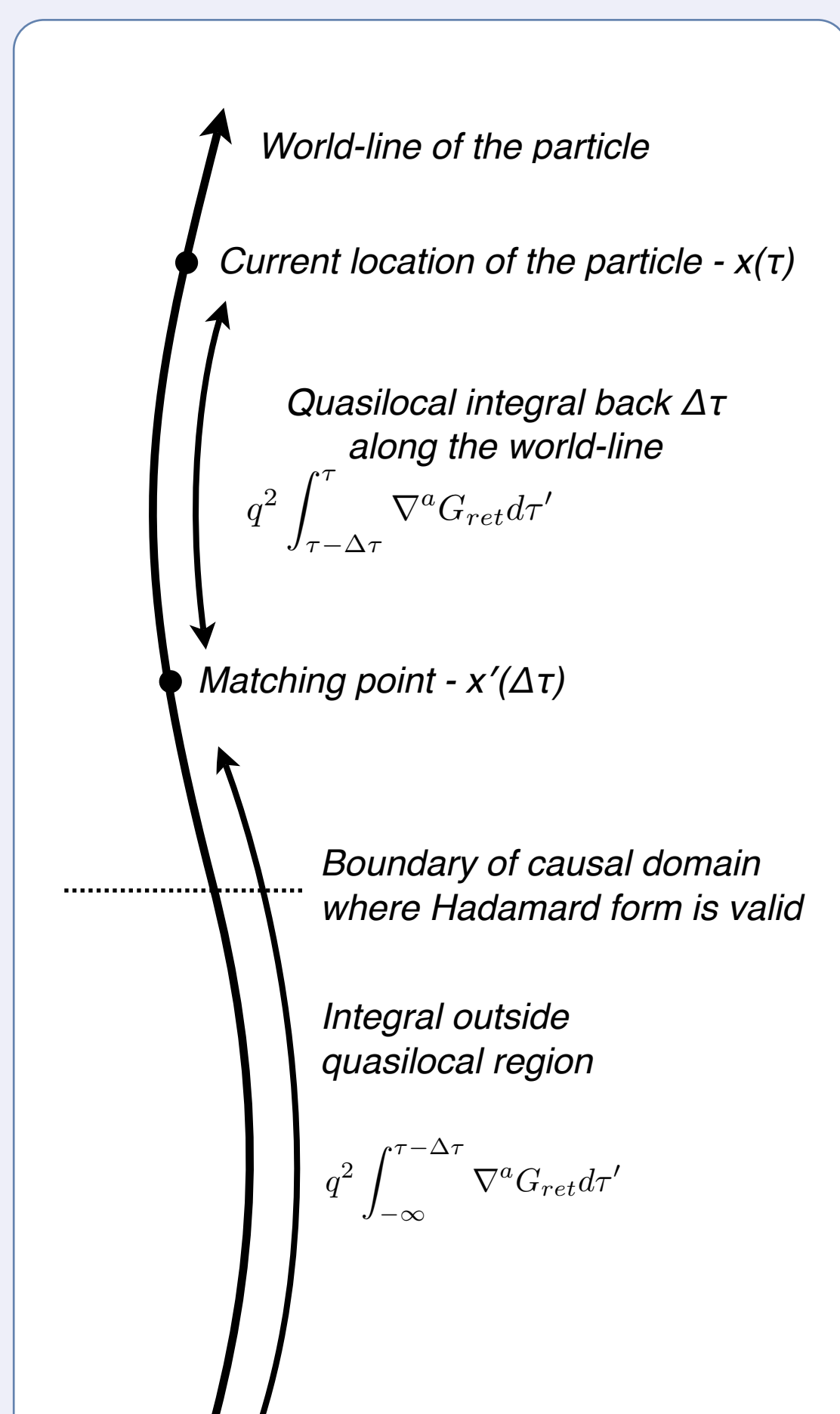


Fig (1): Splitting the integral over the past world-line of the particle into a quasilocal contribution and a contribution for the more distant past

Calculating $V(x, x')$

Within the Hadamard approach, the symmetric biscalar $V(x, x')$ is expressed in terms of a covariant series expansion (i.e. an expansion in increasing powers of σ^a):

$$V(x, x') = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} v_{\alpha_1 \dots \alpha_p}(x) \sigma^{\alpha_1}(x, x') \dots \sigma^{\alpha_p}(x, x')$$

where the $v_{\alpha_1 \dots \alpha_p}$ are calculated from a set of recursion relations [1,3]. Since $\sigma^a = -(\tau - \tau') u^a$ we can easily perform the integration over τ' to obtain an expression for the quasi-local contribution to the self-force in terms of a series expansion in powers of the geodesic distance to the matching point, $\Delta\tau$. In vacuum space-times, this expression is

$$f_{QL}^a = q^2 \left(\frac{1}{4!} v_{\beta\gamma\delta\epsilon} g^{\beta a} u^\gamma u^\delta u^\epsilon \Delta\tau^4 - \frac{1}{5!} \left(\frac{1}{2} v_{\gamma\delta\epsilon\zeta;\beta} - 2v_{\beta\gamma\delta\epsilon;\zeta} \right) g^{\beta a} u^\gamma u^\delta u^\epsilon u^\zeta \Delta\tau^5 + O(\Delta\tau^6) \right)$$

The coefficients in this expansion are given in terms of Riemann tensor components (and its derivative) at the point x , and can therefore be calculated for any vacuum space-time with relative ease. Moreover, the explicit dependence on the 4-velocity u^a means that this can be easily calculated for any particle motion, or even for general motion.

Specific Example - Circular Orbit in Schwarzschild

Using the expression given above, the quasilocal contribution to the self-force can be computed for any geodesic motion in any vacuum space-time. As an example, we take the case of the scalar particle in a circular geodesic orbit around a Schwarzschild black hole of mass M . In order to calculate the quasilocal contribution to the self-force, we simply calculate the components of the Riemann tensor and its derivatives and hence calculate the coefficients, $v_{\alpha_1 \dots \alpha_p}$ appearing in our expansion. This, along with the expressions for the 4-velocity of an object in a circular orbit gives us the result

$$f_{QL}^r = -\frac{3q^2 M^2 (r-2M) (53M^3 + 54rM^2 - 81Mr^2 + 20r^3)}{11200 (r-3M)^2 r^{11}} \Delta\tau^5 + O(\Delta\tau^7)$$

$$f_{QL}^\phi = \frac{9q^2 M^2 (r-2M) (3r-5M)}{2240r^{10} (r-3M)} \sqrt{\frac{M}{r-3M}} \Delta\tau^4 + O(\Delta\tau^6)$$

$$f_{QL}^t = -\frac{3q^2 M^2 (r-2M) (5r-M)}{2240r^9 (r-3M)} \sqrt{\frac{r}{r-3M}} \Delta\tau^4 + O(\Delta\tau^6)$$

Truncation Error

Our result for the quasilocal contribution to the self-force is a series expansion in powers of the matching point $\Delta\tau$. Clearly, by cutting off the expansion at a certain power, we are introducing some truncation error into the result. It is therefore desirable to take the series to as high an order as possible. However, as it will always be necessary to truncate at some point, we would like to estimate this truncation error. This may be done by finding the ratio of the highest order term to the sum of all the terms up to that order, i.e.

$$\epsilon \equiv \frac{f_{QL}^a[n]}{\sum_{i=0}^n f_{QL}^a[i]}$$

where $f_{QL}^a[n]$ is the n th order term in the series. The adjacent plots indicate how this relative error grows with $\Delta\tau$.

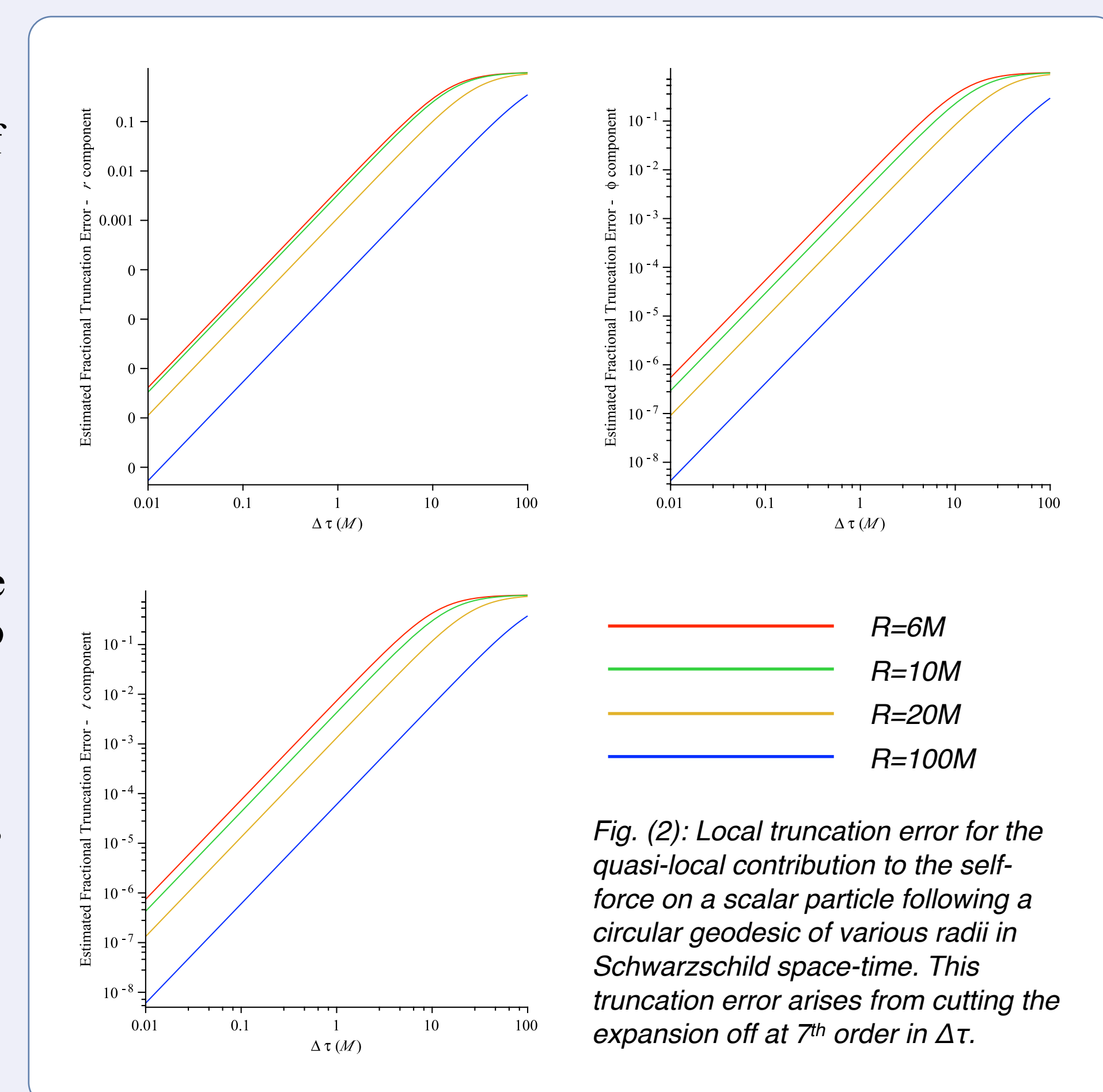


Fig. (2): Local truncation error for the quasi-local contribution to the self-force on a scalar particle following a circular geodesic of various radii in Schwarzschild space-time. This truncation error arises from cutting the expansion off at 7th order in $\Delta\tau$.

References and Further Reading

- [1] Adrian C. Ottewill and Barry Wardell, Phys. Rev. D 77 (2008) 104002.
- [2] T. C. Quinn, Phys. Rev. D 62 (2000) 064029.
- [3] Y. Décanini and A. Folacci, Phys. Rev. D 73 (2006) 044027.

